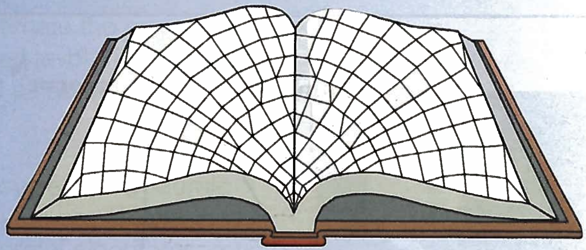
A Finite Element Primer for Beginners: The Basics



Tarek I. Zohdi
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BOOK REVIEW

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108 pages, soft cover, £50 (List Price)
Contents: Preface; 1: Weighted Residuals and Galerkin's Method for a Generic 1-D Problem; 2: A Model Problem: 1-D Elastostatics; 3: A Finite Element Implementation in One Dimension; 4: Accuracy of the Finite Element Method; 5: Element-by-Element Iterative Solution Schemes; 6: Weak Formulations in Three Dimensions; 7: A Finite Element Implementation in Three Dimensions; 8: Accuracy of the Finite Element Method; 9: Time-Dependent Problems; 10: Summary and Advanced Topics; Appendix A: Elementary Mathematical Concepts; Appendix B: Basic Continuum Mechanics; Appendix C: Convergence of Recursive Iterative Schemes.

As the title implies (in three different ways!), this nice little book is an elementary introduction to the Finite Element Method (FEM). I like short books like this, which introduce the essence of a topic to the reader in a hundred or so pages. Another such book which I read many years ago is "A Multigrid Tutorial" by W.L. Briggs, from 1987, which contains 90 pages, and is a delightful introduction to the Multigrid method. Then in 2000, a second edition of that book was published, with two additional authors, and with about twice as many pages. The whole charm of the original book was gone! The original simple and concise exposition has drowned in a sea of details. I am not even sure if I would have bothered with it back then, had the second edition fallen into my hands first. Of course, now the original version is out of print...

The book by Zohdi, which has a printed version and an electronic version, discusses the FEM for 1D and 3D problems in linear elasticity. Everything is written concisely, and many details are omitted, but this fits well the lean format of the Springer Briefs series to which this book belongs. The book looks attractive and is easy to read.

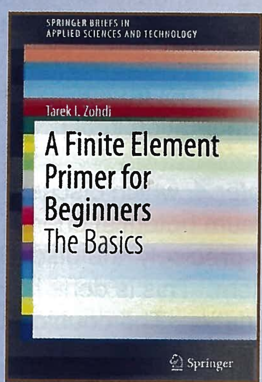
Chapter 1 introduces the ideas behind Weighted Residual methods and the Galerkin method. The motivation given here to Galerkin's method, which I have not seen in other books, is particularly nice. First, we approximate the exact solution by a series of the form

$$u^N = \sum_{i=1}^{N} a_i \phi_i(x) \,, \qquad (1)$$

where the functions ϕ_i are the chosen basis functions which span the approximation space. Now, we wish the approximate solution u^N to be the projection of the exact solution on the approximation space, since in this way we will get the minimal error. See *Figure 1*.

by
DAN GIVOLI
Technion — Israel Institute of Technology
givolid@technion.ac.il

Tarek Zohdi



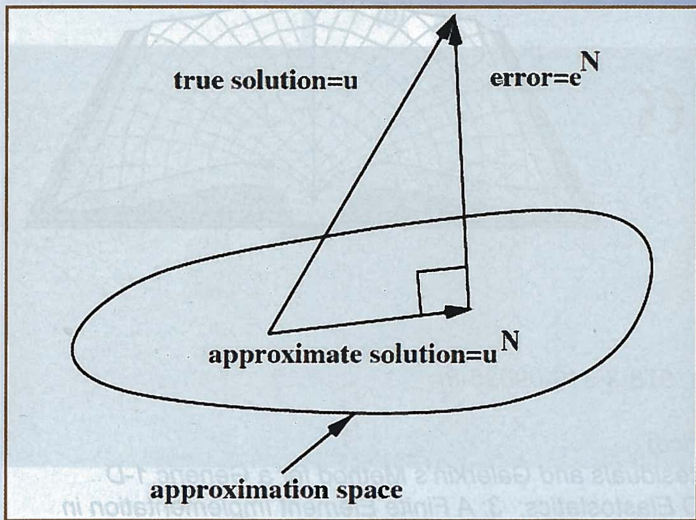


Figure 1:
Orthogonality of the approximation error.
This is Fig. 1.1 from the book.

Therefore, we want the error to be orthogonal to the approximation space, i.e.,

$$\int_{\Omega} e(x)\phi_i(x)dx=0, \qquad (2)$$

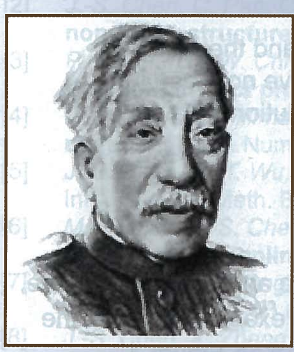
where Ω is the computational domain, and e is the error. Unfortunately, eq. (2) is useless as a practical requirement, since we do not know the error e! So instead of the error, we will take the next best thing to the error, namely the *residual*, which is a quantity we can actually compute. Thus, we replace (2) with the requirement

$$\int_{\Omega} r(x)\phi_i(x)dx=0, \qquad (3)$$

where $r(x)$ is the residual of the differential equation. Substituting the expression for the residual $r(x)$ and using (1) give us (after some additional manipulations) the Galerkin method.

Of course, in the Galerkin method the error *is* orthogonal to the approximation space after all, although not in the L_2 inner-product as implied by (2), but in the energy inner-product. This is not pointed out in *Chapter 1*, and wisely so, since the reader is not yet ready to digest all this. The orthogonality in the energy norm is proved later, in *Chapter 4*.

The exposition described above is abstract, and therefore will be easily accessible to a reader who is mathematically oriented. A reader who is less so, or has never heard of basic functional analysis concepts, such as the projection of a function on a space of functions, would find the exposition a bit harder to absorb. It would have been a good idea to include such concepts in Appendix A. Of course, with an appropriate guidance in class, this should not pose any difficulty.



Boris Galerkin

Another nice discussion is included in *Chapter 2*, on the weak form of the 1D problem. The author explains the matter of smoothness, and the terms “strong” and “weak”. This explanation is missing from most books on FEM. The book from which I first studied FEM as a student proved that the strong and weak forms of the problem are equivalent, and I remember wondering in what way the strong form was “stronger” than the weak form if they were indeed equivalent.

Chapter 2 ends with some comments on material nonlinearity. This provides a glimpse into nonlinear FE formulations, while the book generally covers only the linear case. *Chapter 3* discusses the FE implementation in 1D. Simple examples which can be solved “manually” are given in order to demonstrate the techniques. Section 3.9.4 briefly talks about the saving in storage as the advantage in the sparse structure of the global stiffness matrix. The saving in computing time, which is generally regarded as more critical, is not mentioned here, but is mentioned in Chapter 7, in the context of the FE implementation in 3D.

Chapter 4, which discusses the accuracy of FEM, includes a proof of the principle of minimum potential energy, as the infrastructure used for the Rayleigh-Ritz method. More precisely, the discussion connects between this principle and the weak form of the problem. This is done here in two ways: by an elementary calculation, and by the calculus of variation. The double derivation is nice, as in most books only one of them is provided, if at all.

Chapter 4 also explains, in a simple and clear way, how one can estimate the error constant C and find the mesh parameter h which would provide a desired level of accuracy. In addition it gives a nice mathematical basis to the use of the residual as an indicator in local adaptive refinement schemes.

The Lord Rayleigh

Chapter 5 discusses iterative schemes for the solution of the linear system of algebraic equations, namely steepest descent (SD), conjugate gradients (CG), an pre-conditioned CG. The SD and CG methods are first derived and then written as formal algorithms, ready for implementation.

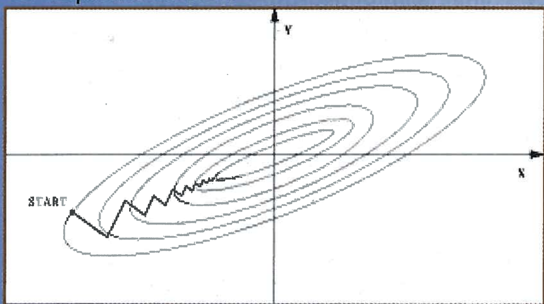
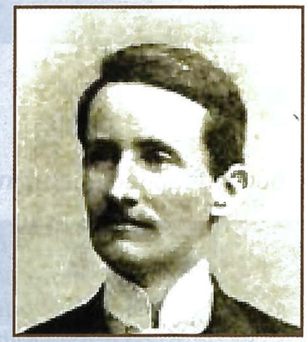


Figure 2:
Typical progress of the minimization in the SD method.
Taken from the thesis of T. Hjorteland, University of Oslo, 1999.

Chapters 6-8 are the 3D analogs of chapters 2-4. Special topics which appear here for the first time include the principle of complementary energy (Chapter 6), where the discussion is limited to the continuous level, and a *posteriori* recovery methods (Chapter 8), briefly presenting the basic idea.



Walther Ritz

Chapter 9 deals with time-dependent problems in linear elasticity. The discretization process is presented in a non-standard way: first the PDE is discretized in time, and only then the resulting semi-discrete equations are discretized in space by FEs. Of course, the final result is the same as in the standard approach. The time-stepping method presented is the Generalized Trapezoidal (GT) method for a first order system. To this end, the elastodynamics equations are cast in first order form, with the displacements and velocities as independent unknowns. This is equivalent to the standard way in which GT is applied to the first-order system

$$\begin{bmatrix} 0 & M \\ I & 0 \end{bmatrix} \begin{Bmatrix} \dot{d} \\ \dot{v} \end{Bmatrix} + \begin{bmatrix} K & 0 \\ 0 & -I \end{bmatrix} \begin{Bmatrix} d \\ v \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \qquad (4)$$

with obvious notation. Forward Euler (explicit) and Backward Euler are presented as special cases.

One may wonder whether it would not have been better to introduce the Newmark method, which is a more standard time-stepping method for problems in elastodynamics. I support the author's choice in this case. The advantage of GT is that it is very simple to present and understand, while introducing the Newmark method would be much more involved. We have to remember the purpose of this book and its elementary style. The author does direct the reader to several other books for further reading.

Leonhard Euler's image on a Swiss Franc bill

Chapter 10 briefly discusses concepts related to domain decomposition and parallel computing. Three appendices follow, which cover concepts from mathematics, continuum mechanics and iterative schemes.

Unfortunately, the book contains many typographical errors, but the reader can avoid confusion by consulting the errata page posted on **http://cmrl.berkeley.edu/zohdipaper/ERRATA-FOR-ZOHDI-FEM-BOOK.pdf** . Hopefully a second and corrected edition will be published soon.

In summary, this is a nice little elementary book on FEM, which has an attractive and non-intimidating form for the student beginning her/his FEM studies, or for individuals whose area of interest is different but desire to get a taste of what FEM is all about.